SAARLAND UNIVERSITY
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Mathematics for Computer Scientists 1, WS 2018/19
Sheet 12

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1. Let a, b be positive numbers and define the function $f : \mathbb{R} \to \mathbb{R}$ by the formula

$$f(x) = \begin{cases} |x|^a \sin\left(\dfrac{1}{|x|^b}\right), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

definiert.

(a) Prove that f is differentiable at all points except 0 and compute its derivative.

(b) For which values of a and b is f also differentiable at the point 0? When, in this case, is its derivative $f' : \mathbb{R} \to \mathbb{R}$

(i) continuous,

(ii) not continuous but bounded,

(iii) not bounded?

2. (a) Sketch the graph of the function $f : \mathbb{R} \to \mathbb{R}$ mit

$$f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-a|},$$

where a is a positive constant, and show that the maximum value of f is

$$\frac{2+a}{1+a}.$$

(b) Let p be a polynomial of degree n with critical points -1, 1, 2, 3 and 4. The corresponding values of p are 6, 1, 2, 4 and 3 and the coefficient of x^n is 1. Sketch the graph of p, distinguishing between the cases n even and n odd.

3. This exercise is to be solved using the following result which is proved in lectures.

Let $f : I \to J$ be a continuous, bijective function with continuous inverse $f^{-1} : J \to I$, where I, J are open intervals. Suppose that f is differentiable at the point a.

f^{-1} is differentiable at the point b if and only if $f'(a) \neq 0$. In this case

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

(a) Let n be a natural number. Define

$$g_n(x) = x^{\frac{1}{n}}, \qquad x \in \mathbb{R},$$

if n is odd and

$$g_n(x) = x^{\frac{1}{n}}, \qquad x \in [0, \infty),$$

if n is even.

(i) Show that g_n is differentiable for $x \neq 0$ and $g_n'(x) = \frac{1}{n} x^{\frac{1}{n}-1}$.

(ii) Deduce that

$$\frac{\mathrm{d}}{\mathrm{d}x} x^q = q x^{q-1}$$

for all *positive* rational numbers q.

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(b) The inverses of

$$\sin(\cdot) : [-\pi/2, \pi/2] \to [-1, 1], \qquad \cos(\cdot) : [0, \pi] \to [-1, 1],$$
$$\tan(\cdot) : (-\pi/2, \pi/2) \to \mathbb{R}$$

are denoted by

$$\arcsin(\cdot) : [-1, 1] \to [-\pi/2, \pi/2], \qquad \arccos(\cdot) : [-1, 1] \to [0, \pi],$$
$$\arctan(\cdot) : \mathbb{R} \to (-\pi/2, \pi/2).$$

Sketch the graphs of these functions and show that

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}, \quad \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}, \qquad x \in (-1, 1),$$

and

$$\arctan'(x) = \frac{1}{1+x^2}, \qquad x \in \mathbb{R}.$$