UNIVERSITÄT DES SAARLANDES
Fachrichtung 6.1 (Mathematik)
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Mathematics for Computer Scientists 1, WS 2017/18
Examination preparation

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1. Find functions with the following properties and justify your answers.

a) $f : [\frac{1}{2}, \infty) \to [-2, 2]$ is injective and strictly monotone decreasing.

b) $f : [0, 1) \to [-1, 1]$ is surjective and monotone increasing.

c) $f : \mathbb{N} \to \mathbb{R}$ is bounded from above, but not from below.

2. Which of the following functions $f : \mathbb{R} \to \mathbb{R}$ is injective, surjective, bijective? (Justify your answers.) Compute $f([-1, 1])$ and $f^{-1}([-1, 1])$ in each case.

(i) $f_1(x) = \begin{cases} x, & x \notin \mathbb{Z}, \\ x - 1, & x \in \mathbb{Z}. \end{cases}$

(ii) $f_2(x) = \begin{cases} x, & x \notin \mathbb{Z}, \\ x^2, & x \in \mathbb{Z}. \end{cases}$

3. Prove the following assertions by mathematical induction.

(i) $7 | (3^{2n+1} + 2^{n+2})$ for each natural number n.

(ii) $n\sqrt{n} > n + \sqrt{n}$ for each natural number $n \geq 4$.

(iii) $\displaystyle\sum_{k=1}^{n} \frac{1}{(k+3)(k+4)} = \frac{n}{4(n+4)}$ for each natural number n.

4. Define relations $\sim_a, \ldots, \sim_e$ on $\mathbb{R}$ by

$$\begin{aligned} x \sim_a y \quad &\Leftrightarrow \quad x \neq y, \\ x \sim_b y \quad &\Leftrightarrow \quad x \leq y, \\ x \sim_c y \quad &\Leftrightarrow \quad x.y \geq 0, \\ x \sim_d y \quad &\Leftrightarrow \quad x \geq y^2, \\ x \sim_e y \quad &\Leftrightarrow \quad x + y \text{ is a whole number.} \end{aligned}$$

Are these relations reflexive, connex, symmetric, asymmetric, antisymmetric and/or transitive? Are they equivalence relations and/or partial orders?

5. Find $[6533]^{-1}$ in $\mathbb{Z}_{7039}$, $[64]^{-1}$ in $\mathbb{Z}_{135}$ and $[543626]^{-1}$ in $\mathbb{Z}_{5436261}$.

6. Compute the solution sets of the following simultaneous equations.

(i)
$$x \equiv 1 (\text{mod } 5),\quad x \equiv 2 (\text{mod } 7),\quad x \equiv 3 (\text{mod } 11).$$

(ii)
$$x \equiv 2 (\text{mod } 3),\quad x \equiv 1 (\text{mod } 4),\quad x \equiv 0 (\text{mod } 7).$$

7. Find all complex solutions to the following equations.

a) $3z^2 + z = 0$

b) $\cos z = 0$

c) $\sinh z = 0$

d) $\tan z = 1$

e) $\cos z = -\frac{5}{4}$

f) $z + \bar{z} = 1$

g) $(1 - \mathrm{i})z^2 = 1 + 7\mathrm{i}$

h) $(1 - \mathrm{i})z^2 = (1 + \mathrm{i})z$

i) $z^3 = 1$

j) $(z^2 - 1)^3 = 8z^3$

k) $\mathrm{e}^z = 1$

l) $\mathrm{e}^{\mathrm{i}z} + 4\mathrm{e}^{-\mathrm{i}z} = 4$

8. Which of the following series are convergent?

a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1 + \mathrm{e}^n}{2^n}$,

b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{n^2 + n^3}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{4n^3 + 6n + 12}{\sqrt{n^8 + n^2}}$,

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$,

e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$,

f) $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-2n}$.

9. Find the radius of convergence for the following power series.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$,

b) $\displaystyle\sum_{n=1}^{\infty} n^{626} x^n$,

c) $\displaystyle\sum_{n=1}^{\infty} n! x^{n^2}$,

d) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{3n4^n} x^n$.

10. Compute the following limits.

a) $\displaystyle\lim_{n\to\infty} \sqrt{n^2 + 3n + 1} - n$,

b) $\displaystyle\lim_{n\to\infty} \frac{(\sqrt{n^5} + 2n) \cdot \left(\sin^2\left(\frac{1}{n}\right) + 1\right)}{(n+1)^2 \sqrt[3]{1 + 2n}}$,

c) $\displaystyle\lim_{x\to 0} \frac{\log(\cos(x))}{\sin(x)}$,

d) $\displaystyle\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$,

e) $\displaystyle\lim_{x\to 1} \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 4x + 3}$,

f) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{2}{x} + \frac{3}{x^2}\right)^{7x}$.

11. Sketch the graphs of the following functions $f : \mathbb{R} \setminus \{-1, 1, 2\} \to \mathbb{R}$.

a) $f(x) = x^2 - x$,

b) $f(x) = \dfrac{x}{x^2 - 1}$,

c) $f(x) = x^5 + x + 1$,

d) $f(x) = \dfrac{x^2 - 8}{(x-2)^2}$.

12. Sketch the graphs of the following functions $f : \mathbb{R} \to \mathbb{R}$ and determine where these functions are differentiable.

a) $$f(x) = \begin{cases} -x, & x \le -1, \\ x^2, & -1 < x < 1, \\ 2x - 1, & x \ge 1. \end{cases}$$

b) $$f(x) = \begin{cases} -x - 3, & x < -1, \\ 2x, & -1 < x < 1, \\ -x + 3, & x > 1, \\ -2, & x \in \{-1, 1\}. \end{cases}$$

13. Compute the Maclaurin series of the functions given by the following formulae and find their radius of convergence.

a) $\dfrac{1}{1 - x^2}$,

b) $\dfrac{x}{1 + 8x^3}$,

c) e^{4x^2},

d) $\dfrac{1}{2 - 4x}$,

e) $\dfrac{1 - \cos(4x)}{2x}$,

f) $\dfrac{1 - \cos(x^2)}{x^4}$.

14. Define the function $f : \mathbb{R} \setminus \{\frac{2}{3}\} \to \mathbb{R}$ by

$$f(x) = \frac{1}{2 - 3x}.$$

Show that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = \frac{n!\, 3^n}{(2 - 3x)^n}$$

for $n \in \mathbb{N}$. Compute the Taylor series of f at 0 and find its radius of convergence.

15. Let the function $f : \mathbb{R} \setminus \{-\frac{9}{2}\} \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{(2x+9)^2}.$$

Show that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = (-1)^n 2^n (n+1)!(2x+9)^{-(n+2)}$$

for $n \in \mathbb{N}$.Compute the Taylor series of f at -4 and find its radius of convergence.

16. Calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log(1+x)).$$

Compute the Taylor series of $\log(1+x)$ at 0 and find its radius of convergence.