SAARLAND UNIVERSITY
Department of Mathematics
Prof. Dr. Mark Groves
MSc Jens Horn

Mathematics for Computer Scientists 2, SS 2018
Sheet 3

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1. a) Show that the improper integrals

$$\int_0^1 \frac{\arcsin t}{\sqrt{1-t^2}} \mathrm{d}t := \lim_{\delta \downarrow 0} \int_0^{1-\delta} \frac{\arcsin t}{\sqrt{1-t^2}} \mathrm{d}t$$

and

$$\int_1^\infty \frac{\log t}{t^2}\, \mathrm{d}t := \lim_{R \to \infty} \int_1^R \frac{\log t}{t^2}\, \mathrm{d}t$$

exist and compute them.

[Hint: Use the substitutions $t = \sin x$ and $t = \mathrm{e}^u$.]

b) Show that the improper integral

$$\int_0^{\pi/2} \frac{1}{1-2\sin t}\, \mathrm{d}t := \lim_{\delta_1 \downarrow 0} \int_0^{\frac{\pi}{6}-\delta_1} \frac{1}{1-2\sin t}\, \mathrm{d}t + \lim_{\delta_2 \downarrow 0} \int_{\frac{\pi}{6}+\delta_2}^{\frac{\pi}{2}} \frac{1}{1-2\sin t}\, \mathrm{d}t$$

does not exist but its Cauchy principal value

$$\lim_{\delta \downarrow 0} \left(\int_0^{\frac{\pi}{6}-\delta} \frac{1}{1-2\sin t}\, \mathrm{d}t + \int_{\frac{\pi}{6}+\delta}^{\frac{\pi}{2}} \frac{1}{1-2\sin t}\, \mathrm{d}t \right)$$

does exist.

[Hint: $\dfrac{\mathrm{d}}{\mathrm{d}t} \dfrac{1}{\sqrt{3}} \log \left(\dfrac{\sin \frac{\pi}{6} - \sin t}{1 - \cos(\frac{\pi}{6} - t)} \right) = \dfrac{1}{1-2\sin t}$ for $t \neq \frac{\pi}{6}$.]

2. The *Gamma function* $\Gamma : (0,\infty) \to \mathbb{R}$ is defined by the formula

$$\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1}\, \mathrm{d}t.$$

a) Show that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

b) Compute $\Gamma(n)$ for $n \in \mathbb{N}$.

c) Use the result $\displaystyle\int_0^\infty \mathrm{e}^{-x^2}\, \mathrm{d}x = \frac{\sqrt{\pi}}{2}$ to compute $\Gamma(\frac{1}{2})$.

[Hint: Use the substitution $t = u^2$.]

3. Demonstrate the rule $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for all vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ geometrically by depicting them as arrows.